Correlations in a Generalized Elastic Model: Fractional Langevin Equation Approach

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The Generalized Elastic Model (GEM) provides the evolution equation which governs the stochastic motion of several many-body systems in nature, such as polymers, membranes, growing interfaces. On the other hand a probe (*tracer*) particle in these systems performs a fractional Brownian motion due to the spatial interactions with the other system's components. The tracer's anomalous dynamics can be described by a Fractional Langevin Equation (FLE) with a space-time correlated noise. We demonstrate that the description given in terms of GEM coincides with that furnished by the relative FLE, by showing that the correlation functions of the stochastic field obtained within the FLE framework agree to the corresponding quantities calculated from the GEM. Furthermore we show that the Fox H-function formalism appears to be very convenient to describe the correlation properties within the FLE approach.

I. INTRODUCTION

Polymers [1–3], elastic chains and membranes [2, 4–7], rough surfaces [8–10] can be described by a continuum elastic model which accounts for their general stochastic behavior. This model, named Generalized Elastic Model (GEM), is defined by the following stochastic linear differential equation in partial derivatives [11],

$$\frac{\partial}{\partial t}\mathbf{h}(\vec{x},t) = \int d^d x' \Lambda(\vec{x}-\vec{x}')\frac{\partial^z}{\partial|\vec{x}'|^z}\mathbf{h}(\vec{x}',t) + \boldsymbol{\eta}(\vec{x},t).$$
(1)

The stochastic field $\mathbf{h}$ is D-dimensional and defined in the $d-$dimensional infinite space. The white noise appearing in (1) satisfies the fluctuation-dissipation (FD) relation, i.e.

$$\langle \eta_j(\vec{x},t)\eta_k(\vec{x}',t')\rangle = 2k_B T \Lambda(\vec{x}-\vec{x}')\delta_{j\,k}\delta(t-t').$$
(2)

($j,k \in [1,D]$) where $\Lambda(\vec{r})$ corresponds to the hydrodynamic friction kernel which couples different sites in $\vec{x}$ and $\vec{x}'$ through a fluid-mediated interaction. The internal elastic coupling is instead embodied by the fractional derivative defined via its Fourier transform by [12]

$$\mathcal{F}_{\vec{q}}\left\{\frac{\partial^z}{\partial|\vec{x}|^z}\right\} \equiv -|\vec{q}|^z.$$
(3)

Another common definition is given in term of the Laplacian Δ as [13]: $\frac{\partial^z}{\partial|\vec{x}|^z} := -(-\Delta)^{z/2}$.

The systems whose the dynamical behavior is described by the GEM (1) can be divided into two different classes, according to the type of hydrodynamic interactions that characterize them: *long ranged* or *local*.

A. Long ranged hydrodynamic interactions

This situation occurs when the friction kernel is defined by a general expression like

$$\Lambda(\vec{r}) \sim \frac{1}{|\vec{r}|^\alpha},$$
(4)

where $a \ll |\vec{r}| \ll L$, L corresponds, for example, to the screening length or the maximum size of the system, and a is the smallest length scale up to which the continuum description furnished by (1) keeps its validity. Whenever our analysis will require a regularization at small and/or long distances, the largest and smallest length scales will then come into play as the integral's upper and/or lower cut-offs.

The hydrodynamic interactions are often represented by the equilibrium average of the Oseen tensor, which in an embedding d_e-dimensional space ($d_e \geq 3$) reads, according to (4): $\Lambda(\vec{r}) \sim |\vec{r}|^{2-d_e}$ [1, 14]. The following systems belong to this class.

- *Fluid membranes.* The height of a fluctuating membrane is represented by the scalar quantity $h(\vec{x},t)$, i.e. $D = 1$ [2, 5–7]. The point $\vec{x}$ on the planar surface implies $d = 2$, which in turn gives $d_e = D + d = 3$ and $\alpha = 1$. Since for small deformations ($\left|\vec{\nabla}h\right| \ll 1$) the bending free energy is $\propto (\Delta h)^2$ [15], $z = 4$ in (1).

- *Semiflexible polymers.* $\mathbf{h}$ stands for the 3 spatial coordinates of a polymeric segment (*bead*) while x is the strand's 1-dimensional internal coordinate (*curvilinear abscissa*): $D = 3$, $d = 1$. The embedding dimension in this case coincides with D ($d_e = 3$), yielding $\alpha = 1$ [2, 3]. The bending elastic energy associated with the chain's deformation implies $z = 4$ [16] as in the previous example.

- *Flexible polymers.* In this systems $\mathbf{h}$ and x still correspond to the position and the curvilinear ascissa of the monomer in the polymeric chain: $D = 3$, $d = 1$. However the beads interaction is just represented by an harmonic coupling ($z = 2$) and the Zimm's equilibrium approximation of the Oseen tensor in Θ solvent gives $\alpha = 1/2$ [1, 17].

B. Local hydrodynamic interactions

This kind of systems present no fluid-mediated interactions, namely $\Lambda(\vec{r}) = \delta^d(\vec{r})$. This can be attributed either to the large screening among the elementary components of the system or to an interaction which is purely mechanical. Examples are:

- *Rouse polymers.* Here, as in the case of (semi)flexible chains, **h** stands for the bead's 3-dimensional position, and x for the bead's position along the chain [1, 18]. The hookean interaction gives $z = 2$.

- *Single file system.* The system of N hard-core rods diffusing on a line without overlapping is known in literature as single file (see [19] and references therein). Recently it has been shown that the system's dynamics can be reduced, within a very good approximation, to a 1-dimensional harmonic chain problem (*harmonization*) where $h(x, t)$ is the position of the x-th particle on the substrate at time t.

- *Fluctuating interfaces.* In this systems h plays the role of a scalar field (mostly the height of a rough surface in d dimension) which is subjected to a non-standard elastic force embodied by the fractional derivative of order z [8, 9]. This is actually the generalization of the Edwards-Wilkinson equation for the fluctuating profile of a granular surface, for which $d = 2$, $z = 2$ [4]. In systems such as crack propagation [20] and contact line of a liquid meniscus [21] $d = 1$, and the restoring forces are characterized by $z = 1$.

- *Solid surfaces.* If h is a step, namely a line boundary at which the surface changes height by one or more atomic units [10], the value of z in eq.(1) is found to be $z = 2, 3$ or 4 $(d = 1)$ according to the character of the atomic diffusion: periphery, terrace or attachment-detachment diffusion respectively.

- *Diffusion-noise equation.* In this case h represents the density field on a d-dimensional surface $\vec{x}$ while the diffusion operator sets $z = 2$ [22].

The values of the parameters related to each of the models formerly listed are summarized in table I.

In [11] we addressed the question of the motion of a tracer (*probe*) particle in the systems whose dynamics obeys to (1). Although the whole system dynamics is Markovian, the particle placed at a position $\vec{x}$ undergoes a subdiffusive motion on the score of persistent memory effects due to the spatial correlations with the rest of the system. Roughly speaking, a tracer particle experiences two kinds of interactions: the first one is the coupling with the surrounding heat bath, whose mathematical expression is furnished by the Langevin random force $\eta(\vec{x}, t)$ and the corresponding FD relation (2). The second interaction is inherent to the system: the probe particle is coupled with the rest of the system through both the hydrodynamic term (4) and the fractional Laplacian (3). This "internal" coupling originates the spatial correlations responsible for the tracer's long-ranged non-

System	D	d	z	α
Fluid membranes [2, 5–7]	1	2	4	1
Semiflexible polymers [2, 3]	3	1	4	1
Flexible polymers [1, 17]	3	1	2	1/2
Crack propagation [20]	1	1	1	-
Liquid meniscus [21]	1	1	1	-
Rouse polymers [1, 18]	3	1	2	-
Single file systems [19]	1	1	2	-
Fluctuating interfaces [4, 8, 9]	1	any	any	-
Solid surfaces [10]	1	1	2,3,4	-
Diffusion-noise equation [22]	1	any	2	-

TABLE I. Values of the parameters D, d, z and α charachterizing the GEM (1) for the systems listed in the Introduction. Note that for systems whose the hydrodynamic interaction is local the value of α is absent.

Markovian memory effects. On the other hand, the non-Markovianity of the probe particle's anomalous dynamics leads to the description in terms of fractional Brownian motion (FBM) [23] which obeys a fractional Langevin equation (FLE) [19, 24]. Within this framework the strong internal interactions are mimicked by the colored noise term and the fractional derivative, connected by the generalized fluctuation-dissipation relation. In this article we will show how the representation of the tracer's stochastic evolution given in terms of FLE offers the same level of accuracy furnished by (1). Indeed any kind of physical statistical observable can be evaluated starting from (1) as well as from the corresponding FLE.

This paper is outlined as follows: in Sec.II we start from the GEM (1) deriving the expressions for the h-autocorrelation function (and the corresponding mean square displacement) in the case of $z > d$, $z < d$ and $z = d$ $((d-1)/2 < \alpha < d)$. In Section III, starting from the expression (1), we draw the FLE equation for the tracer particle placed at position $\vec{x}$ when $z > d$. In Section IV we derive the properties of the noise appearing in the FLE and we demonstrate the validity of the fluctuation-dissipation relation for the probe particle. Section V will be devoted to the h-correlation function arised from the FLE framework and we furnish its general expression in terms of the Fox functions. In particular we apply the developed formalism to systems such as fluid membranes, proteins and fluctuating interfaces, recovering results previously derived in literature. In Appendix A we report the calculations for the hydrodynamics term in the limiting case $\alpha = d$. In Appendix B we provide a demonstration of the appeareance of the Fox functions in our analysis, while in Appendix C are listed the main properties of the Fox functions that we make use of through our calculations. Lastly, Appendix E concerns with the derivation of the Generalized Langevin Equation for the inter-monomeric distance in a 3-dimensional Rouse chain, according to the procedure outlined in [19].

II. h-AUTOCORRELATION FUNCTION FOR THE GENERALIZED ELASTIC PROCESS

We start from the equation (1). Defining Fourier transform of the stochastic process in space and time as

$$\mathbf{h}(\vec{q},\omega) = \int_{-\infty}^{+\infty} d^d x \int_{-\infty}^{+\infty} dt\, \mathbf{h}(\vec{x},t)\, e^{-i(\vec{q}\cdot\vec{x}-\omega t)}, \quad (5)$$

we find that the general solution of (1) can be expressed in the Fourier space in the following form

$$\mathbf{h}(\vec{q},\omega) = \frac{\boldsymbol{\eta}(\vec{q},\omega)}{-i\omega + \Lambda(\vec{q})\,|\vec{q}|^z}, \quad (6)$$

where $\Lambda(\vec{q})$ is the Fourier transform of hydrodynamic friction kernel,

$$\Lambda(\vec{q}) \begin{cases} = \frac{(4\pi)^{d/2}}{2^\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}|\vec{q}|^{\alpha-d} = A|\vec{q}|^{\alpha-d} & \frac{d-1}{2} < \alpha < d \\ \sim \frac{2\pi^{d/2}}{\Gamma(d/2)}\ln\left(\frac{1}{|\vec{q}|a}\right) & \alpha = d. \end{cases} \quad (7)$$

To get eq.(7) we use the expression for the d-dimensional Fourier transform of the isotropic function $\phi(|\vec{r}|)$ [25]

$$\int_{-\infty}^{+\infty} d^d r\, e^{-i\vec{q}\cdot\vec{r}}\phi(|\vec{r}|) = (2\pi)^{d/2}|\vec{q}|^{1-d/2}\int_0^{+\infty} d|\vec{r}|\,|\vec{r}|^{d/2}J_{d/2-1}(|\vec{q}||\vec{r}|)\phi(|\vec{r}|), \quad (8)$$

where $J_{d/2-1}$ represents the Bessel function of fractional order $d/2-1$. The limiting case $\alpha = d$ requires a regularization of the integral in the $\vec{q}$-space: this has been done introducing the cut-off a. Note that if we resort to the different definition $\Lambda(\vec{r}) \sim \frac{1}{a^d+|\vec{r}|^d}$, we get the same asymptotic expansion (for small $|\vec{q}|$) obtained in (7) for $\alpha = d$; in this case however, the complete results for $d = 1, 2$ and 3 are presented in Appendix A. The case $\alpha = (d-1)/2$ which requires the infrared cut-off L won't be treated in the following.

In the local hydrodynamic situation $\Lambda(\vec{r}) = \delta(\vec{r})$, which corresponds to take $\Lambda(\vec{q}) = 1$ in (6). Therefore the calculations for local hydrodynamic systems can be either performed starting from (6) where $\Lambda(\vec{q}) = 1$, or simply setting $A = const$ and $\alpha = d$ in the corresponding long-ranged hydrodynamic expressions: this substitution, *which is not to be intended as a limit*, allows to easily shift from power-law to local hydrodynamics throughout the following analysis. Indeed it corresponds to a formal procedure to obtain the results for local hydrodynamic systems, starting from the equivalent quantities elaborated for long ranged hydrodynamic models. As a consequence, the case $\alpha = d$ and the ensuing logarithmic behavior (7), does not have to be confused with the systems with local hydrodynamic interactions. We then define the h-autocorrelation function as $\langle \delta h(\vec{x},t)\,\delta h(\vec{x},t')\rangle =$

$\langle [h_j(\vec{x},t) - h_j(\vec{x},0)][h_j(\vec{x},t') - h_j(\vec{x},0)]\rangle$. Because of the isotropy of the system henceforth we will drop the index j. Since the Fourier transform of the noise correlation function (2) gets the form $\langle \eta_j(\vec{q},\omega)\eta_k(\vec{q}',\omega')\rangle = 2k_B T(2\pi)^{d+1}\Lambda(\vec{q})\,\delta_{j\,k}\delta(\omega+\omega')\,\delta^d(\vec{q}+\vec{q}')$, after a bit of algebra we derive

$$\langle\delta h(\vec{x},t)\,\delta h(\vec{x},t')\rangle = k_B T\times \\ \times \int_{-\infty}^{+\infty}\frac{d^d q}{(2\pi)^d}\frac{g(|\vec{q}|,t)+g(|\vec{q}|,t')-g(|\vec{q}|,|t-t'|)}{|\vec{q}|^z} \\ g(|\vec{q}|,t) = 1 - e^{-\Lambda(\vec{q})|\vec{q}|^z t}. \quad (9)$$

When $(d-1)/2 < \alpha < d$ the general expression of (9) has the following form:

$$\langle\delta h(\vec{x},t)\,\delta h(\vec{x},t')\rangle = K[f(t)+f(t')-f(|t-t'|)]. \quad (10)$$

Three different situations can occur.

i) $z < d$ $(\alpha > d-z)$. The integrals in (9) are divergents when $|\vec{q}| \to \infty$: hence once again we are compelled to introduce the cut-off a. Performing the integrations yields

$$K = \frac{2k_B T\pi^{d/2}}{(2\pi)^d\Gamma(d/2)}\frac{A^\beta}{\alpha+z-d} \\ f(t) = -\beta\left[A\left(\frac{\pi}{a}\right)^{\alpha+z-d}\right]^{-\beta} - \gamma\left(-\beta, A\left(\frac{\pi}{a}\right)^{\alpha+z-d}t\right)t^\beta, \quad (11)$$

where

$$\beta = \frac{z-d}{\alpha+z-d} \quad (12)$$

and the function $\gamma(a,x)$ is defined in [31]. It is straightforward to verify that in this case the mean square displacement gets to a constant value, $\langle\delta^2 h(\vec{x},t)\rangle \to \frac{4k_B T\pi^{d/2}}{(2\pi)^d\Gamma(d/2)(d-z)}\left(\frac{\pi}{a}\right)^{d-z}$. Physically this means that the system is overconnected and asymptotically any probe remains trapped around its initial position.

ii) $z = d$. The integrals in (9) exhibit a logarithmic divergence. After the regularization through the insertion of a the result is achieved substituting in (10)

$$K = \frac{2k_B T\pi^{d/2}}{(2\pi)^d\Gamma(d/2)\alpha} \\ f(t) = E_1\left(A\left(\frac{\pi}{a}\right)^\alpha t\right) + \ln\left[A\left(\frac{\pi}{a}\right)^\alpha t\right] + C, \quad (13)$$

where C is a constant value that we set to 0 and $E_1(x)$ denotes the exponential integral [31]. In this case the system attains an asymptotic logarithmic diffusion: $\langle\delta^2 h(\vec{x},t)\rangle \to \frac{4k_B T\pi^{d/2}}{(2\pi)^d\Gamma(d/2)\alpha}\ln\left[A\left(\frac{\pi}{a}\right)^\alpha t\right]$. This is a borderline case, where the probe is not completely free to diffuse away from its initial position: the ensuing erratic motion is then logarithmic.

iii) $z > d$. This is the most interesting situation: in this case the integrals can be performed effortlessly to give

$$K = \frac{2k_B T \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \frac{A^\beta \Gamma(1-\beta)}{z-d}$$
$$f(t) = t^\beta. \tag{14}$$

Therefore, the tracer particle placed at a given $\vec{x}$ performs a subdiffusive fractional Brownian motion (FBM) [23] given by $\langle \delta^2 h(\vec{x},t) \rangle = 2Kt^\beta$. In the language of fluctuating interfaces an interface is called *rough* in this case [9].

In the following Sections we will focus on the situation *iii*) for which $z > d$.

III.   FRACTIONAL LANGEVIN EQUATION

Starting from the solution (6) we now discuss the derivation of the fractional Langevin equation for the probe particle placed at a given position $\vec{x}$.

Let us first multiply both sides of (6) by $K^+(-i\omega)^\beta$. As one can see from the derivation presented below, this is the only choice leading to the FLE which obeys the FD relation. According to [11], choosing another arbitrary power instead of $(-i\omega)^\beta$ would lead to another equation among the family of Generalized Fractional Langevin Equations (GFLE), but without FD relation fulfiled. Thus eq.(6) becomes

$$K^+(-i\omega)^\beta h(\vec{q},\omega) = \frac{K^+(-i\omega)^\beta}{-i\omega + A|\vec{q}|^{z+\alpha-d}} \eta(\vec{q},\omega). \tag{15}$$

K^+ is a constant that we introduce in order to fulfill the fluctuation-dissipation relation: indeed this physical constraint must be satisfied regardless of the description made of the tracer's dynamics, i.e. both by the Markovian Langevin description given in (1) and by the fractional Langevin representation that we are aimed at deriving.

Equation(15) can be rewritten as

$$K^+(-i\omega)^\beta h(\vec{q},\omega) = K^+ \eta(\vec{q},\omega) \Phi(\vec{q},\omega), \tag{16}$$

where we introduce the function $\Phi(\vec{x},t)$ whose Fourier transform in space and time reads

$$\Phi(\vec{q},\omega) = \frac{(-i\omega)^\beta}{-i\omega + A|\vec{q}|^{z+\alpha-d}}, \tag{17}$$

Now, we simply make an inverse Fourier transform of Eq.(16) in space and time. In the right hand side we get a new noise term $\zeta(\vec{x},t)$, which is the convolution of $\Phi(\vec{x},t)$ with the white Gaussian noise $\eta(\vec{x},t)$, i.e.

$$\zeta(\vec{x},t) = K^+ \int_{-\infty}^{\infty} d^d x' \int_{-\infty}^{\infty} dt' \, \eta(\vec{x}-\vec{x}',t-t') \, \Phi(\vec{x}',t'). \tag{18}$$

To transform the left hand side of Eq.(16), we introduce the Caputo derivative with lower bound equal to $-\infty$, which for a "sufficiently well-behaved" function $\phi(t)$ is defined as follows [13, 32],

$$D_+^\beta \phi(t) = \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{t} dt' \frac{1}{(t-t')^\beta} \frac{d}{dt'} \phi(t'), \ 0 < \beta < 1, \tag{19}$$

and whose Fourier transform reads as

$$\int_{-\infty}^{\infty} dt \, e^{i\omega t} D_+^\beta \phi(t) = (-i\omega)^\beta \phi(\omega). \tag{20}$$

Thus, we get finally the fractional Langevin equation for the stochastic field $h(\vec{x},t)$ [6, 11, 19, 24, 26],

$$K^+ D_+^\beta h(\vec{x},t) = \zeta(\vec{x},t). \tag{21}$$

From Eq.(21) and the definition (19), the requirement of the validity of the fluctuation-dissipation relation reads

$$\langle \zeta(\vec{x},t) \zeta(\vec{x},t') \rangle = k_B T \frac{K^+}{\Gamma(1-\beta)|t-t'|^\beta}, \tag{22}$$

which relates the autocorrelation function of the noise (standing in the right hand side of eq.(21)) to the damping kernel (determined by the fractional derivative (19)) in the left hand side of eq.(21). From this requirement the value of K^+ will be set. This will be done in the next Section. It is important to remark that the random field (18) is still Gaussian, since it is the linear combination of the Gaussian noise $\eta(\vec{x},t)$ but is also power-law correlated in time according to the FD relation (22), hence is a fractional Gaussian noise (fGn).

IV.   FRACTIONAL GAUSSIAN NOISE CORRELATION FUNCTION

We want to evaluate the two-point two-time correlation function of the fGn appearing in (21). From (2) and from the definition (18) we obtain

$$\langle \zeta(\vec{x},t) \zeta(\vec{x'},t') \rangle = 2k_B T K^{+2} \times$$
$$\times \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \varphi(\vec{x}-\vec{x'},\omega) \tag{23}$$

where

$$\varphi(\vec{x}-\vec{x'},\omega) =$$
$$\int d\vec{x_1} d\vec{x_2} \Lambda(\vec{x_1}-\vec{x_2}) \Phi(\vec{x}-\vec{x_1},\omega) \Phi(\vec{x'}-\vec{x_2},-\omega). \tag{24}$$

To calculate (24) we make use of the hydrodynamic term's Fourier transform (7) and of (17): after straightforward manipulations it takes the form

$$\varphi\left(\vec{x}-\vec{x'},\omega\right)=\frac{A}{(2\pi)^{d/2}}|\omega|^{2\beta}\left|\vec{x}-\vec{x'}\right|^{1-d/2}\times$$
$$\times\int_0^\infty d\,|\vec{q}|\,\frac{|\vec{q}|^{\alpha-d/2}J_{d/2-1}\left(|\vec{q}||\vec{x}-\vec{x'}|\right)}{\omega^2+A^2|\vec{q}|^{2(z+\alpha-d)}},\tag{25}$$

which, after a change of variable, becomes

$$\varphi\left(\vec{x}-\vec{x'},\omega\right)=\frac{A^{\frac{2z-d-2}{2(z+\alpha-d)}}}{(2\pi)^{d/2}}|\omega|^{\frac{-2\alpha-d+2}{2(z+\alpha-d)}}\left|\vec{x}-\vec{x'}\right|^{1-d/2}\times$$
$$\times\int_0^\infty dy\,\frac{y^{\alpha-d/2}}{1+y^{2(z+\alpha-d)}}J_{d/2-1}\left(\frac{y|\omega|^{1/(z+\alpha-d)}|\vec{x}-\vec{x'}|}{A^{1/(z+\alpha-d)}}\right)\tag{26}$$

To proceed further, at this stage we employ the formalism of the Fox $H-$functions. These functions, introduced by Fox in 1961 [33], are special functions of a very general nature which allow us to present the results in a universal and elegant fashion. For a general theory on the $H-$functions we address the reader to the monograph of Mathai and Saxena [34] and to ref. [35]. As interesting applications of $H-$functions we could mention an exactly solvable model of linear viscoelastic behavior [36], the $H-$function representation of non-Debye relaxation [37, 38] and of the solution of the space-time fractional diffusion equations [39, 40]. We present the defintion and the basic properties of the $H-$functions in Appendices B and C, respectively. Moreover in Appendix B it is shown that the function $1/(1+y^{2(z+\alpha-d)})$ appearing in the expression (26) can be cast in term of a Fox function:

$$\frac{1}{1+y^\gamma}=\frac{1}{\gamma}H_{1\,1}^{1\,1}\left[y\left|\begin{array}{c}\left(0,\frac{1}{\gamma}\right)\\\left(0,\frac{1}{\gamma}\right)\end{array}\right.\right]\tag{27}$$

where we introduced the short notation

$$\gamma=2(z+\alpha-d)\tag{28}$$

After this substitution (26) reads

$$\varphi\left(\vec{x}-\vec{x'},\omega\right)=\frac{A^{\beta+\frac{d-2}{\gamma}}}{(2\pi)^{d/2}\gamma}|\omega|^{\frac{-2\alpha-d+2}{\gamma}}\left|\vec{x}-\vec{x'}\right|^{1-d/2}\times$$
$$\times\int_0^\infty dy\,y^{\alpha-d/2}J_{d/2-1}(ky)H_{1\,1}^{1\,1}\left[y\left|\begin{array}{c}\left(0,\frac{1}{\gamma}\right)\\\left(0,\frac{1}{\gamma}\right)\end{array}\right.\right]\tag{29}$$

with $k=(|\omega|/A)^{2/\gamma}\left|\vec{x}-\vec{x'}\right|$. Using the property (C4) the integral can be evaluated to give

$$\varphi\left(\vec{x}-\vec{x'},\omega\right)=\frac{A^{\beta+\frac{d-2}{\gamma}}}{(2\pi)^{d/2}\gamma}|\omega|^{\frac{-2\alpha-d+2}{\gamma}}\left|\vec{x}-\vec{x'}\right|^{1-d/2}\times$$
$$\frac{2^{\alpha-d/2}}{k^{\alpha+1-d/2}}H_{3\,1}^{1\,2}\left[\frac{2}{k}\left|\begin{array}{cc}\left(1-\frac{\alpha}{2},\frac{1}{2}\right)&\left(0,\frac{1}{\gamma}\right)&\left(1-\frac{\alpha+2-d}{2},\frac{1}{2}\right)\\&\left(0,\frac{1}{\gamma}\right)&\end{array}\right.\right],\tag{30}$$

which, thanks to (C2) and (C3), gets the form

$$\varphi\left(\vec{x}-\vec{x'},\omega\right)=\frac{A^{\beta+\frac{d-2}{\gamma}}}{2(2\pi)^{d/2}\gamma}|\omega|^{\frac{-2\alpha-d+2}{\gamma}}\left|\vec{x}-\vec{x'}\right|^{1-d/2}\times$$
$$H_{1\,3}^{2\,1}\left[\frac{k}{2}\left|\begin{array}{cc}&\left(\frac{4z+2\alpha-3d-2}{2\gamma},\frac{1}{\gamma}\right)\\\left(\frac{d-2}{4},\frac{1}{2}\right)&\left(\frac{4z+2\alpha-3d-2}{2\gamma},\frac{1}{\gamma}\right)&\left(\frac{2-d}{4},\frac{1}{2}\right)\end{array}\right.\right].\tag{31}$$

Applying again the property (C3), the noise correlation function (23) can be written as

$$\langle\zeta(\vec{x},t)\zeta(\vec{x'},t')\rangle=\frac{2k_BTK^{+2}A^{2\beta-1}}{2^{\alpha+d}\gamma\pi^{d/2+1}}\left|\vec{x}-\vec{x'}\right|^\alpha\int_0^{+\infty}d\omega\times$$
$$\times\cos(\omega(t-t'))H_{1\,3}^{2\,1}\left[\frac{k}{2}\left|\begin{array}{cc}&\left(\beta,\frac{1}{\gamma}\right)\\\left(\frac{-\alpha}{2},\frac{1}{2}\right)&\left(\beta,\frac{1}{\gamma}\right)&\left(\frac{2-\alpha-d}{2},\frac{1}{2}\right)\end{array}\right.\right],\tag{32}$$

where β and γ are given by (12) and (28) respectively. The integral in (32) can be solved by referring to the property (C5) of the Fox function: the final expression for the noise correlation function then reads

$$\langle\zeta(\vec{x},t)\zeta(\vec{x'},t')\rangle=\frac{k_BTK^{+2}A^{2\beta-1}}{2^{\alpha+d}\pi^{(d+1)/2}}\frac{|\vec{x}-\vec{x'}|^\alpha}{|t-t'|}\times$$
$$\times H_{3\,3}^{2\,2}\left[\frac{2}{A|t-t'|}\left(\frac{|\vec{x}-\vec{x'}|}{2}\right)^{\frac{\gamma}{2}}\left|\begin{array}{ccc}\left(\frac{1}{2},\frac{1}{2}\right)&\left(\beta,\frac{1}{2}\right)&\left(0,\frac{1}{2}\right)\\\left(-\frac{\alpha}{2},\frac{\gamma}{4}\right)&\left(\beta,\frac{1}{2}\right)&\left(\frac{2-\alpha-d}{2},\frac{\gamma}{4}\right)\end{array}\right.\right].\tag{33}$$

The former expression is the central result of this paper. It states that the fGn entering the FLE (21) is not only correlated in time, as required by the physical constraint (22), but it is also space-correlated. This means that the space correlations appearing in (1), which are embodied by the hydrodynamic term as well as by the fractional Laplacian, are translated into space-time correlations of the noise in the FLE dynamical representation.

We recall now that the coefficient K^+ in (33) and (22) is still undefined, our aim is to set its expression. For this purpose we calculate the autocorrelation function of the noise, i.e. we set $\vec{x}\equiv\vec{x'}$ in (33). To do so, we first expand the Fox function for small argument according to (C7) and restric ourselves to the main term which diverges at $\vec{x}\equiv\vec{x'}$:

$$H_{3\,3}^{2\,2}\left[\frac{2}{A|t-t'|}\left(\frac{|\vec{x}-\vec{x'}|}{2}\right)^{\frac{\gamma}{2}}\left|\begin{array}{ccc}\left(\frac{1}{2},\frac{1}{2}\right)&\left(\beta,\frac{1}{2}\right)&\left(0,\frac{1}{2}\right)\\\left(-\frac{\alpha}{2},\frac{\gamma}{4}\right)&\left(\beta,\frac{1}{2}\right)&\left(\frac{2-\alpha-d}{2},\frac{\gamma}{4}\right)\end{array}\right.\right]=$$
$$\frac{2^{2+\alpha}\sqrt{\pi}A^{1-\beta}}{\gamma}\frac{\Gamma(\beta)}{\Gamma(d/2)}\frac{|t-t'|^{1-\beta}}{|\vec{x}-\vec{x'}|^\alpha}.\tag{34}$$

We then plug such expression in (33), achieving the following final form for the fractional Gaussian noise autocorrelation function :

$$\langle\zeta(\vec{x},t)\zeta(\vec{x},t')\rangle=\frac{k_BT}{|t-t'|^\beta}K^{+2}\frac{A^\beta2^{2-d}}{\pi^{d/2}\gamma}\frac{\Gamma(\beta)}{\Gamma(d/2)}.\tag{35}$$

Thus the comparison between (35) and (22) yields the value of K^+, namely

$$K^+ = (4\pi)^{d/2-1}\gamma \sin(\pi\beta)\frac{\Gamma(d/2)}{A^\beta}. \quad (36)$$

V. TWO-POINT TWO-TIME h-CORRELATION FUNCTION

In this Section we address the problem of the evaluation of the h-correlation function within the framework of the FLE. Indeed, any kind of statistical observable can be expressed in term of correlation functions, whose analytical expression can be derived either starting from the eq.(1) or from eq.(21). Although the representation of the system dynamics can be different, the correlation functions must coincide, since the observable physical properties do not have to depend on the chosen description. Therefore we can furnish the general expression of the correlation function starting from the solution of (21). The solution of (21) in the Fourier-Fourier space reads

$$h(\vec{q},\omega) = \frac{\zeta(\vec{q},\omega)}{K^+(-i\omega)^\beta}, \quad (37)$$

and consequently the two-point two-time correlation function is

$$\langle h(\vec{x},t)h(\vec{x'},t')\rangle = $$
$$\frac{1}{K^{+2}}\int\frac{d\omega\,d\omega'}{4\pi^2}e^{-i\omega t}e^{-i\omega' t'}\frac{\langle\zeta(\vec{x},\omega)\zeta(\vec{x'},\omega')\rangle}{(-i\omega)^\beta(-i\omega')^\beta}. \quad (38)$$

Using (32), the Fox function expression for (38) reads

$$\langle h(\vec{x},t)h(\vec{x'},t')\rangle = \frac{2k_B T A^{2\beta-1}}{2^{\alpha+d}\gamma\pi^{d/2+1}}\left|\vec{x}-\vec{x'}\right|^\alpha \int_0^{+\infty}d\omega\times$$
$$\times\frac{\cos(\omega(t-t'))}{\omega^{2\beta}}H_{1\,3}^{2\,1}\left[\frac{k}{2}\left|\begin{array}{c}\left(\beta,\frac{1}{\gamma}\right)\\ \left(\frac{-\alpha}{2},\frac{1}{2}\right)\ \left(\beta,\frac{1}{\gamma}\right)\ \left(\frac{2-\alpha-d}{2},\frac{1}{2}\right)\end{array}\right.\right]. \quad (39)$$

The previous expression constitutes the elementary component starting with whom, any kind of physical observable is constructed. Nonetheless, the integral appearing in (39) cannot be solved as we performed in (32). The reason is that such integral is divergent in the limit $\omega\to 0$, as it is apparent by an expansion of the Fox function at small argument, i.e., recalling that $k=\left(\frac{|\omega|}{A}\right)^{2/\gamma}\left|\vec{x}-\vec{x'}\right|$ and using (C7), one has

$$H_{1\,3}^{2\,1}\left[\frac{k}{2}\left|\begin{array}{c}\left(\beta,\frac{1}{\gamma}\right)\\ \left(\frac{-\alpha}{2},\frac{1}{2}\right)\ \left(\beta,\frac{1}{\gamma}\right)\ \left(\frac{2-\alpha-d}{2},\frac{1}{2}\right)\end{array}\right.\right]\simeq$$
$$\frac{2\pi}{\Gamma(d/2)\cos(\beta/2)}\left(\frac{2}{|\vec{x}-\vec{x'}|}\right)^\alpha\left(\frac{A}{|\omega|}\right)^{1-\beta}. \quad (40)$$

It follows from (40) and (39) that the integrand in (39) diverges as $\sim|\omega|^{-(1+\beta)}$. Therefore, in order be measurable, any physical observable has to be arranged in such a way that the divergence will be eliminated. In the following subsections we will analyze some specific case of statistical quantities constructed as linear combination of (39) and compare them with the corresponding quantities arising from (6).

A. Mean Square Displacement

The mean square displacement of the probe particle has been defined in secton II as the limit $t\to t'$ of the autocorrelation function (9); in term of elementary component (39) it can be expressed as

$$\langle\delta^2 h(\vec{x},t)\rangle = 2\left[\langle h^2(\vec{x},0)\rangle - \langle h(\vec{x},t)h(\vec{x},0)\rangle\right]. \quad (41)$$

Using the expansion (40) the expression (41) can be casted as

$$\langle\delta^2 h(\vec{x},t)\rangle = \frac{8k_B T(4\pi)^{-d/2}A^\beta}{\gamma\Gamma(d/2)\cos(\pi\beta/2)}\int_0^{+\infty}d\omega\frac{1-\cos(\omega t)}{(\omega)^{1+\beta}}. \quad (42)$$

Solving the integral we get the expression (14) obtained previously from the GEM (1).

B. Dynamic structure factor of fluid membranes

In ref. [7] Zilman and Granek derived the short length and short time behavior of the dynamic structure factor of the fluid membranes. As mentioned in the Introduction, the fluid membrane dynamics correspond to take $D=1$, $d=2$ and $z=4$ in (1), moreover the hydrodynamic friction kernel is expressed as

$$\Lambda(\vec{r}) = \frac{1}{8\pi\xi\,|\vec{r}|} \quad (43)$$

which gives $\alpha=1$ in (4), ξ is the solvent viscosity. Note that unlike the original model we set the bending modulus $\kappa=1$. According to (7) and the definition (43) the constant A is found to be $A=1/(4\xi)$.

The quantity which has been studied in [7] is the two-point correlation function

$$\langle\left[h(\vec{x},t)-h(\vec{x'},0)\right]^2\rangle = 2\left[\langle h^2(\vec{x},0)\rangle - \langle h(\vec{x},t)h(\vec{x'},0)\rangle\right] \quad (44)$$

which is shown to be linked to the membrane's dynamic structure factor. The last expression can thus be recast in term of the quantity (39): using the expansion (40) and the numerical values of d, z, α and A we find

$$\langle\left[h\left(\vec{x},t\right)-h\left(\vec{x'},0\right)\right]^2\rangle =$$
$$\frac{2\,k_BT}{3\pi(4\xi)^{2/3}}\int_0^{+\infty}d\omega\left\{\frac{1}{\omega^{5/3}}-\cos\left(\omega t\right)\frac{(4\xi)^{1/3}}{8\pi\omega^{4/3}}\left|\vec{x}-\vec{x'}\right|\times\right.$$
$$\left.\times\,H_{1\,3}^{2\,1}\left[\left|\vec{x}-\vec{x'}\right|\frac{(4\xi\omega)^{1/3}}{2}\left|\begin{array}{cc}&\left(\frac{2}{3},\frac{1}{6}\right)\\\left(\frac{-1}{2},\frac{1}{2}\right)&\left(\frac{2}{3},\frac{1}{6}\right)\quad\left(-\frac{1}{2},\frac{1}{2}\right)\end{array}\right.\right]\right\}$$
(45)

After changing variable $\left(y=\left|\vec{x}-\vec{x'}\right|(4\xi\omega)^{1/3}/2\right)$ and making use of the property (C3) the following general simpler expression is achieved for the two-point correlation function:

$$\langle\left[h\left(\vec{x},t\right)-h\left(\vec{x'},0\right)\right]^2\rangle=\frac{k_BT}{2\pi}\int_0^{+\infty}dy\frac{\left|\vec{x}-\vec{x'}\right|^2}{y^3}\times$$
$$\times\left\{1-\frac{\cos\left(\left(\frac{2y}{\left|\vec{x}-\vec{x'}\right|}\right)^3\frac{t}{4\xi}\right)}{4\pi}H_{1\,3}^{2\,1}\left[y\left|\begin{array}{cc}&\left(\frac{5}{6},\frac{1}{6}\right)\\\left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)\quad\left(0,\frac{1}{2}\right)\end{array}\right.\right]\right\}$$
(46)

Although eq.(46) furnishes a compact analitycal expression for the correlation function at any time and any distance $\left|\vec{x}-\vec{x'}\right|$, the integral cannot be computed esplicitely since it displays a logarithmic divergence in the limit $y\to 0$. Again this can be seen by expanding the cosine and the Fox function in (46) to the second order:

$$H_{1\,3}^{2\,1}\left[y\left|\begin{array}{cc}&\left(\frac{5}{6},\frac{1}{6}\right)\\\left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)\quad\left(0,\frac{1}{2}\right)\end{array}\right.\right]\simeq 4\pi-2\pi y^2+\pi y^4.\quad(47)$$

We first analyze the limit $t=0$ which corresponds to the static correlator $\langle\left[h\left(\vec{x},0\right)-h\left(\vec{x'},0\right)\right]^2\rangle$, describing the membrane's roughness. In this case we immediately get from (46)

$$\langle\left[h\left(\vec{x},0\right)-h\left(\vec{x'},0\right)\right]^2\rangle=\frac{k_BT}{2\pi}\int_0^{+\infty}dy\frac{\left|\vec{x}-\vec{x'}\right|^2}{y^3}\times$$
$$\times\left\{1-\frac{1}{4\pi}H_{1\,3}^{2\,1}\left[y\left|\begin{array}{cc}&\left(\frac{5}{6},\frac{1}{6}\right)\\\left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)\quad\left(0,\frac{1}{2}\right)\end{array}\right.\right]\right\}.$$
(48)

On the score of the previous discussion, using the expansion (47) the above expression can be approximated as

$$\langle\left[h\left(\vec{x},0\right)-h\left(\vec{x'},0\right)\right]^2\rangle\simeq\frac{k_BT}{4\pi}\left|\vec{x}-\vec{x'}\right|^2\times$$
$$\times\left(\int_{\left|\vec{x}-\vec{x'}\right|/L}^1\frac{dy}{y}-\frac{1}{2}\int_0^1 dy\,y\right)$$
(49)

where L stands for the long scale cut-off representing the membrane's size. The underlying assumption in (49) is that the major contribution to the integral in (48) comes from $y\leq 1$; this, in turn, justifies the lower cut-off appearing in the first integral of (49): indeed the minimum relaxation frequency of the membrane's bending modes will be given by $\omega_0=\left(\frac{2}{L}\right)^3\frac{1}{4\xi}$ which corresponds to $y_0=\left|\vec{x}-\vec{x'}\right|/L$. Hence solving (49) we obtain the expression which coincides with that found in [7] for the static correlator, i.e.

$$\langle\left[h\left(\vec{x},0\right)-h\left(\vec{x'},0\right)\right]^2\rangle\simeq$$
$$\frac{k_BT}{4\pi}\left|\vec{x}-\vec{x'}\right|^2\left[\ln\left(\frac{L}{\left|\vec{x}-\vec{x'}\right|}\right)-0.25\right]$$
(50)

valid whenever $\left|\vec{x}-\vec{x'}\right|\ll L$. The value of -0.25 of the correction term can be improved in a regular way; see Appendix D.

For the dynamic correlator we can still consider the main contribution to the integral arising from $y\leq 1$ and use (47), then

$$\langle\left[h\left(\vec{x},t\right)-h\left(\vec{x'},0\right)\right]^2\rangle\simeq\frac{k_BT}{2\pi}\left|\vec{x}-\vec{x'}\right|^2\times$$
$$\times\left[\int_0^1 dy\frac{1-\cos\left(\left(\frac{2y}{\left|\vec{x}-\vec{x'}\right|}\right)^3\frac{t}{4\xi}\right)}{y^3}+\frac{1}{2}\int_0^1 dy\frac{\cos\left(\left(\frac{2y}{\left|\vec{x}-\vec{x'}\right|}\right)^3\frac{t}{4\xi}\right)}{y}\right].$$
(51)

Changing the variable back to ω, for times in the intermediate range $4\xi\left(\frac{\left|\vec{x}-\vec{x'}\right|}{2}\right)^3\ll t\ll 4\xi\left(\frac{L}{2}\right)^3$ we can safely replace the upper bound of the integrals to ∞ achieving

$$\langle\left[h\left(\vec{x},t\right)-h\left(\vec{x'},0\right)\right]^2\rangle\simeq\frac{k_BT}{2\pi}\left|\vec{x}-\vec{x'}\right|^2\times$$
$$\times\left[\frac{(4\xi)^{-2/3}}{3}\left(\frac{4}{\left|\vec{x}-\vec{x'}\right|}\right)^2\int_0^\infty d\omega\frac{1-\cos(\omega t)}{\omega^{5/3}}+\frac{1}{6}\int_0^\infty dy\frac{\cos(\omega t)}{\omega}\right].$$
(52)

Thus we can compute the first integral and introduce the infrared cut-off ω_0 to regularize the second, therefore the final expression for the correlation function reads

$$\langle\left[h\left(\vec{x},t\right)-h\left(\vec{x'},0\right)\right]^2\rangle\simeq\frac{k_BT}{2\pi}\left|\vec{x}-\vec{x'}\right|^2\times$$
$$\times\left[\frac{\Gamma(1/3)}{(4\xi)^{2/3}}\frac{t^{2/3}}{\left|\vec{x}-\vec{x'}\right|^2}-\frac{Ci\left(\frac{2t}{L^3\xi}\right)}{6}\right].$$
(53)

where Ci is the cosine integral [31]. The result (53) matches and amends the corresponding expression furnished in [7].

C. Donor-Acceptor correlation function in proteins

In Refs [26, 27] it has been shown that the dynamics of the donor-acceptor (D-A) distance within a protein can be mapped into the motion of a fictitious particle obeying a FLE with fractional derivative of order 1/2, in presence of an harmonic potential whose frequency ω_0^2

could be phenomenologically inferred *a posteriori* from the experimental data. The detected quantity was the autocorrelation function of the D-A distance $\boldsymbol{\Delta}_{D-A}(t)$ that was shown to display an asymptotic Mittag-Leffler decay in accordance with the FLE prescription. In order to recover the experimental results, in Refs [28, 29] and [30] the authors used a respectively continuous and discrete Rouse model accounting for the protein conformational dynamics: this, in turn, corresponds to take $D = 3$, $d = 1$, $z = 2$ and $\Lambda(x - x') = \delta^d(x - x')$ in (1) (see Introduction). The simple Rouse model was shown to reproduce the Mittag-Leffler decay of the $\boldsymbol{\Delta}_{D-A}$ autocorrelation function for large t and, on the other hand, it was shown to lead to the correct 1/2-FLE for the D-A distance within the framework developed in [11, 19], with the frequency $\omega_0^2 \propto 1/(x_A - x_D)$. We now want to calculate the D-A autocorrelation function without resorting to the derivation of the FLE for the D-A distance (which will be done in Appendix E), but starting from the FLE (21) valid for the monomer placed at position x measured along the protein's profile.

According to [28] we define the $\boldsymbol{\Delta}_{D-A}$ autocorrelation function as

$$\langle \boldsymbol{\Delta}_{D-A}(t) \cdot \boldsymbol{\Delta}_{D-A}(t') \rangle = 3 \langle \Delta_{D-A}(t) \Delta_{D-A}(t') \rangle \quad (54)$$

which can be rewritten in term of the spatial positions of the donor and acceptor sites, $h(x_D, t)$ and $h(x_A, t)$ respectively, as

$$\begin{aligned}
&\langle \boldsymbol{\Delta}_{D-A}(t) \cdot \boldsymbol{\Delta}_{D-A}(t') \rangle = \\
&6 \left(\langle h(x_A, t) h(x_A, t') \rangle - \langle h(x_A, t) h(x_D, t') \rangle \right)
\end{aligned} \quad (55)$$

Therefore, putting in (39) the numerical value of the parameters and changing variable ($y = |x_A - x_D| \sqrt{|\omega|}/2$) we get

$$\begin{aligned}
&\langle \boldsymbol{\Delta}_{D-A}(t) \cdot \boldsymbol{\Delta}_{D-A}(t') \rangle = \frac{3\sqrt{2}\, k_B T}{\pi} |x_A - x_D| \times \\
&\times \int_0^{+\infty} dy \frac{\cos\left(\left(\frac{2y}{|x_A - x_D|} \right)^2 |t - t'| \right)}{y^2} \times \\
&\times \left\{ 1 - \frac{1}{2^{3/2}\sqrt{\pi}} H_{1\,3}^{2\,1} \left[y \, \middle| \, \begin{array}{ccc} & \left(\frac{3}{4}, \frac{1}{4} \right) & \\ \left(0, \frac{1}{2} \right) & \left(\frac{3}{4}, \frac{1}{4} \right) & \left(\frac{1}{2}, \frac{1}{2} \right) \end{array} \right] \right\}.
\end{aligned} \quad (56)$$

Expanding the H-function in the former expression gives

$$H_{1\,3}^{2\,1} \left[y \, \middle| \, \begin{array}{ccc} & \left(\frac{3}{4}, \frac{1}{4} \right) & \\ \left(0, \frac{1}{2} \right) & \left(\frac{3}{4}, \frac{1}{4} \right) & \left(\frac{1}{2}, \frac{1}{2} \right) \end{array} \right] \simeq 2^{3/2}\sqrt{\pi} \left[1 - 2y^2 \right]. \quad (57)$$

which guarantees that the integral in (56) does not present any divergence: we then seek for its explicit solution. First we express the Fox function's Mellin transform following the definition (B1): according to (B2) it is found to be

$$\chi(s) = \frac{2^{3/2}\sqrt{\pi}\,\Gamma(s) \left[\sin\left(\frac{\pi s}{4} \right) + \cos\left(\frac{\pi s}{4} \right) \right]}{2^s} \quad (58)$$

which can be inverted [42] to give

$$\begin{aligned}
&H_{1\,3}^{2\,1} \left[y \, \middle| \, \begin{array}{ccc} & \left(\frac{3}{4}, \frac{1}{4} \right) & \\ \left(0, \frac{1}{2} \right) & \left(\frac{3}{4}, \frac{1}{4} \right) & \left(\frac{1}{2}, \frac{1}{2} \right) \end{array} \right] = \\
&2^{3/2}\sqrt{\pi} e^{-\sqrt{2}y} \left(\sin\sqrt{2}y + \cos\sqrt{2}y \right).
\end{aligned} \quad (59)$$

Plugging (59) in (56) and integrating by parts, we obtain

$$\begin{aligned}
&\langle \boldsymbol{\Delta}_{D-A}(t) \cdot \boldsymbol{\Delta}_{D-A}(t') \rangle = \frac{3\sqrt{2}\, k_B T}{\pi} |x_A - x_D| \times \\
&\times \left[+2C \int_0^\infty dy \sin(Cy^2) e^{-\sqrt{2}y} \left(\sin\sqrt{2}y + \cos\sqrt{2}y \right) + \right. \\
&\left. +2\sqrt{2} \int_0^\infty dy \frac{\sin\sqrt{2}y \cos(Cy^2)}{y} e^{-\sqrt{2}y} - 2C \int_0^\infty dy \sin Cy^2 \right]
\end{aligned} \quad (60)$$

where $C = \frac{4|t-t'|}{|x_A - x_D|^2}$. Let's analyze the three integrals within the square brackets. First we consider the following general expression

$$\int_0^\infty dy\, y e^{-\beta y} \sin(Cy^2) \sin(\beta y) = \sqrt{\frac{\pi}{2C^3}} \frac{\beta}{4} e^{-\frac{\beta^2}{2C}}. \quad (61)$$

We then integrate with respect to β both the RHS and the LHS of (61) to achieve [43]

$$\int_0^\infty dy\, e^{-\beta y} \sin(Cy^2) \left(\sin(\beta y) + \cos(\beta y) \right) = \sqrt{\frac{\pi}{C}} \frac{e^{-\frac{\beta^2}{2C}}}{2^{3/2}} \quad (62)$$

Multiplying both sides by $2C$ and setting $\beta = \sqrt{2}$ eq.(62) yields

$$\begin{aligned}
&2C \int_0^\infty dy \sin(Cy^2) e^{-\sqrt{2}y} \left(\sin(\sqrt{2}y) + \cos(\sqrt{2}y) \right) = \\
&\sqrt{\frac{\pi C}{2}} e^{-\frac{1}{C}}.
\end{aligned} \quad (63)$$

We next consider the second integral in (60), hereby named $I_2(C)$, and differentiate it with respect to C, i.e.

$$\frac{d}{dC} I_2(C) = -\sqrt{\frac{\pi}{C^3}} \frac{e^{-\frac{1}{C}}}{\sqrt{2}}. \quad (64)$$

The former differential equation can be solved thanks to the initial condition [43]

$$I_2(0) = 2\sqrt{2} \int_0^\infty dy \frac{\sin\sqrt{2}y}{y} e^{-\sqrt{2}y} = \frac{\pi}{\sqrt{2}} \quad (65)$$

yielding

$$I_2(C) = \frac{\pi}{\sqrt{2}}\, erf\left[\frac{1}{\sqrt{C}}\right] \tag{66}$$

where erf denotes the error function [31].

The third integral in (60) can be evaluated effortlessly [43]

$$-2C\int_0^\infty dy \sin Cy^2 = -\sqrt{\frac{\pi C}{2}}. \tag{67}$$

Putting (63), (66) and (67) in (60) and substituting the expression of C, the final form of the D-A autocorrelation function is achieved:

$$\langle \boldsymbol{\Delta}_{D-A}(t)\cdot\boldsymbol{\Delta}_{D-A}(t')\rangle =$$
$$\frac{3\sqrt{2}\,k_BT}{\pi}|x_A-x_D|\times\left\{\sqrt{\frac{2\pi|t-t'|}{|x_A-x_D|^2}}\left(e^{-\frac{|x_A-x_D|^2}{4|t-t'|}}-1\right)+\right.$$
$$\left.+\frac{\pi}{\sqrt{2}}\,erf\left[\frac{|x_A-x_D|}{2}\sqrt{\frac{1}{|t-t'|}}\right]\right\} \tag{68}$$

which is exactly the expression found in ref [19]. Moreover (68) recovers the asymptotic decay of the autocorrelation function $C_x(t)$ observed in the experiments [26, 27] that indeed was found to be

$$C_x(t) = \frac{k_BT}{\omega_0^2}E_{1/2,1}\left[-\sqrt{\frac{t}{t_0}}\right] \sim \frac{k_BT}{\omega_0^2\sqrt{\pi}}\sqrt{\frac{t_0}{t}}, \tag{69}$$

where ω_0 is the charachteristic frequency of the potential and $t_0 = \left(\frac{\xi}{\omega_0^2}\right)^2$, with ξ generalized damping coefficient. On the other hand the expression (68) asymptotically attains the form

$$\langle\Delta_{D-A}(0)\Delta_{D-A}(t)\rangle \sim \frac{3k_BT(x_A-x_D)^2}{\sqrt{\pi}}\frac{1}{\sqrt{t}} \tag{70}$$

which matches the former expression once $t_0 = |x_A-x_D|^2$ and $\omega_0 \propto 1/\sqrt{|x_A-x_D|}$ (see Appendix E). We note that, whereas (68) does not furnish a *pure* Mittag-Leffler decay, the comparison with the real experimental data is very good even at short times, as shown in [30]. Moreover, the expression (68) and the corresponding single-file's [19] provide a compact and elegant representation of the D-A distance correlation function, reproducing the results obtained formerly starting from the same Rouse's protein models [28, 30].

D. Correlation function in fluctuating interfaces

In fluctuating interfaces [8–10], often referred as rough surfaces, the height $h(\vec{x},t)$ obeys to the following generalized growth equation (generalized elastic line)

$$\frac{\partial}{\partial t}h(\vec{x},t) = \frac{\partial^z}{\partial|\vec{x}|^z}h(\vec{x},t) + \eta(\vec{x},t). \tag{71}$$

Accordingly $\Lambda\left(\vec{x}-\vec{x'}\right) = \delta^d\left(\vec{x}-\vec{x'}\right)$ while z and d are left unspecified. The growth of the interface is charachterized by two different growth exponents: the dynamical exponent z and the roughness exponent ζ_r. These two exponents control the correlations among the neighboring heights: the correlation between two different sites $\vec{x}$ and $\vec{x'}$ grows according to $\left|\vec{x}-\vec{x'}\right| \propto t^{1/z}$, while the difference between the corresponding heights behaves like $\left|h(\vec{x},t)-h\left(\vec{x'},t\right)\right| \propto \left|\vec{x}-\vec{x'}\right|^{\zeta_r}$. A scaling argument [44] requires that

$$\zeta_r = \frac{z-d}{2} \tag{72}$$

since eq.(71) must be invariant under scale transformation. However we can obtain both the scaling form and the exponents in a rigourous way, by studying the following correlation function

$$\langle[h(\vec{x},t)-h(\vec{x},0)]\left[h\left(\vec{x'},t'\right)-h\left(\vec{x'},0\right)\right]\rangle =$$
$$\langle h(\vec{x},t)h\left(\vec{x'},t'\right)\rangle - \langle h(\vec{x},t)h\left(\vec{x'},0\right)\rangle -$$
$$-\langle h\left(\vec{x'},t'\right)h(\vec{x},0)\rangle + \langle h(\vec{x},0)h\left(\vec{x'},0\right)\rangle. \tag{73}$$

Thanks to (39) the former function reads (here $\alpha = d$, $\gamma = 2z$ and $\beta = (z-d)/z$)

$$\langle[h(\vec{x},t)-h(\vec{x},0)]\left[h\left(\vec{x'},t'\right)-h\left(\vec{x'},0\right)\right]\rangle =$$
$$\frac{k_BT}{4^d\pi^{d/2+1}z}\left|\vec{x}-\vec{x'}\right|^d\int_0^{+\infty}d\omega\,\omega^{2\frac{d-z}{z}}\times$$
$$\times H_{13}^{21}\left[\frac{\left|\vec{x}-\vec{x'}\right|\omega^{\frac{1}{z}}}{2}\left|\begin{array}{c}\left(\frac{z-d}{z},\frac{1}{2z}\right)\\\left(-\frac{d}{2},\frac{1}{2}\right)\ \left(\frac{z-d}{d},\frac{1}{2z}\right)\ \left(1-d,\frac{1}{2}\right)\end{array}\right.\right]\times$$
$$\times[\cos(\omega|t-t'|)-\cos(\omega t)-\cos(\omega t')+1]. \tag{74}$$

Changing variable $\left(y = \left|\vec{x}-\vec{x'}\right|\omega^{\frac{1}{z}}/2\right)$ and using (C3) the expression (74) can be recasted as

$$\langle[h(\vec{x},t)-h(\vec{x},0)]\left[h\left(\vec{x'},t'\right)-h\left(\vec{x'},0\right)\right]\rangle =$$
$$\frac{k_BT}{2^z\pi^{d/2+1}}\left|\vec{x}-\vec{x'}\right|^{z-d}\int_0^{+\infty}dy\times$$
$$\times H_{13}^{21}\left[y\left|\begin{array}{c}\left(\frac{z-1}{2z},\frac{1}{2z}\right)\\\left(\frac{d-z-1}{2},\frac{1}{2}\right)\ \left(\frac{z-1}{2z},\frac{1}{2z}\right)\ \left(\frac{1-z}{2},\frac{1}{2}\right)\end{array}\right.\right]\times$$
$$\times\left[\cos\left((2y)^z\frac{|t-t'|}{|\vec{x}-\vec{x'}|^z}\right)-\cos\left((2y)^z\frac{t}{|\vec{x}-\vec{x'}|^z}\right)-\right.$$
$$\left.-\cos\left((2y)^z\frac{t'}{|\vec{x}-\vec{x'}|^z}\right)+1\right]. \tag{75}$$

Eq.(75) is the generalization of a well-known scaling formula obtained for the two-point two-time correlation function of the Edward-Wilkinson chain [44, 45]. On the other hand the width of a growing surface obeys the *Family-Vicsek* scaling relation [46]:

$$w(L,t) \equiv \sqrt{\frac{1}{L^d} \int_{L^d} d\vec{x} \, [h(\vec{x},t) - \langle h(\vec{x},t) \rangle]^2} \sim L^{\zeta_r} g\left(\frac{t}{L^z}\right) \tag{76}$$

where L is the maximum size of the system (L^d is the total volume) and g is a scaling function. Comparing (76) with (75) we find both the scaling expression and the correct value of the scaling exponents, i.e Eq.(72). Moreover we want to stress that, to the knowledge of the authors, this is the first time that a scaling function appearing in the surface growth is given in an explicit form.

The integrals entering Eqs(73) and (74) can be performed with the use of the H-functions' properties, and the result is expressed in terms of their combination. Here we analyze only the simple case of the Edward-Wilkinson chain, thus we set $z = 2$ and $d = 1$ in (75) [47]. Thanks to (59) the correlation function gets the expression

$$\langle [h(x,t) - h(x,0)] [h(x',t') - h(x',0)] \rangle =$$
$$\frac{k_B T}{\sqrt{2\pi}} |x - x'| \int_0^{+\infty} dy \frac{e^{-\sqrt{2}y}}{y^2} \left[\sin(\sqrt{2}y) + \cos(\sqrt{2}y) \right] \times$$
$$\times \left[\cos\left(Ay^2\right) - \cos\left(By^2\right) - \cos\left(Cy^2\right) + 1 \right]. \tag{77}$$

where $A = 4\frac{|t-t'|}{|x-x'|^2}$, $B = 4\frac{t}{|x-x'|^2}$ and $C = 4\frac{t'}{|x-x'|^2}$. Making use of the integrals (63) and (66) we achieve the final scaling form

$$\langle [h(x,t) - h(x,0)] [h(x',t') - h(x',0)] \rangle =$$
$$\frac{k_B T}{\sqrt{2}} |x - x'| \left[f\left(\frac{t}{|x-x'|^2}\right) + f\left(\frac{t'}{|x-x'|^2}\right) - f\left(\frac{|t-t'|}{|x-x'|^2}\right) \right]$$
$$f(u) = \sqrt{\frac{2u}{\pi}} e^{-\frac{1}{4u}} - \frac{1}{\sqrt{2}} erfc\left(\sqrt{\frac{1}{4u}}\right). \tag{78}$$

It is straightforward to verify that

$$f(u) \begin{cases} \sim 8u^{3/2} e^{-\frac{1}{4u}} & u \to 0 \\ \sim \sqrt{u} & u \to \infty. \end{cases} \tag{79}$$

VI. SUMMARY AND CONCLUDING REMARKS

In this article we have shown how the Markovian representation of the system's dynamics furnished by the GEM (1) is equivalent to the non-Markovian description of the tracer's dynamics given in terms of the FLE (21). Firstly we want stress that the introduced FLE describes the time evolution of the randomm field, which depends not only upon time (as the usual FLE do) but also on space variable. Indeed, although the FLE reproduces the anomalous stochastic motion of the field $h(\vec{x},t)$ at a given position $\vec{x}$ disregarding the remaining systems' dynamics, the internal spatial correlations appear through the noise term which is correlated in time *and* in space. This is the novelty of our approach: strictly speaking we don't loose any information in passing from the Markovian GEM (1) to the non-Markovian FLE (21), on the contrary we rather ease the calculations! Indeed, the appearance of the Fox function in the two-point two-time correlation function, as well as in the noise correlation function, makes the computation of any physical observable relatively easy, if resorting to the few general Fox function's properties. Moreover, as in the case of fluctuating interfaces, the proposed framework allows the expression of the statistical quantities in an explicit analytical and elegant form involving the H-functions.

Appendix A: Hydrodynamic term's Fourier transform: case $\alpha = d$

If the hydrodynamic term is expressed as

$$\Lambda(\vec{r}) = \frac{1}{a^d + |\vec{r}|^d} \tag{A1}$$

its Fourier transform expression reads

$$\Lambda(\vec{q}) = (2\pi)^{d/2} |\vec{q}|^{1-d/2} \int_0^\infty dr \frac{r^{d/2}}{a^d + r^d} J_{d/2-1}(|\vec{q}|r), \tag{A2}$$

where r stands for $|\vec{r}|$ and $J_{d/2-1}$ represents the Bessel function of order $d/2 - 1$. In what follows we will provide the exact result for the 1, 2, and 3-dimensional cases.

i) $d = 1$. In this case the expression (A2) takes the simple form

$$\Lambda(q) = 2 \int_0^\infty dr \frac{\cos(qr)}{a + r}. \tag{A3}$$

Solving Eq.(A3) we get the following result

$$\Lambda(q) = 2 \left[\sin(qa) \left(\frac{\pi}{2} - Si(qa) \right) - \cos(qa) Ci(qa) \right], \tag{A4}$$

where Si and Ci represent respectively the sine and cosine integrals [31] .

ii) $d = 2$. We have

$$\Lambda(q) = 2\pi \int_0^\infty dr \frac{r}{a^2 + r^2} J_0(|\vec{q}|r) \qquad \text{(A5)}$$

which gives [43]

$$\Lambda(\vec{q}) = 2\pi K_0(|\vec{q}|a), \qquad \text{(A6)}$$

where K_0 stands for the modified Bessel function of 0-th order [31].

iii) $d = 3$. Eq.(A2) takes the following form,

$$\Lambda(\vec{q}) = \frac{(2\pi)^{3/2}}{\sqrt{|\vec{q}|}} \int_0^\infty dr \frac{r^{3/2}}{a^3 + r^3} J_{1/2}(|\vec{q}|r), \qquad \text{(A7)}$$

which can be rewritten as

$$\Lambda(\vec{q}) = \frac{4\pi}{|\vec{q}|} \int_0^\infty dr \frac{r}{a^3 + r^3} \sin(|\vec{q}|r). \qquad \text{(A8)}$$

The integral in the previous expression can be splitted into the sum of two contributions, i.e. $\Lambda(\vec{q}) = \frac{4\pi}{|\vec{q}|}(I_1(\vec{q}) - I_2(\vec{q}))$ with

$$\begin{cases} I_1(\vec{q}) = \int_0^\infty dr \frac{\sin(|\vec{q}|r)}{(r-x)(r-x*)} \\ I_2(\vec{q}) = a \int_0^\infty dr \frac{\sin(|\vec{q}|r)}{(r+a)(r-x)(r-x^*)}, \end{cases} \qquad \text{(A9)}$$

where $x = ae^{i\pi/3}$ and $x^* = ae^{-i\pi/3}$. We first study $I_1(\vec{q})$, which can be easily transformed in

$$I_1(\vec{q}) = \frac{1}{ia\sqrt{3}} \left[\int_{-x}^\infty dy \frac{\sin(|\vec{q}|(y+x))}{y} - c.c. \right] \qquad \text{(A10)}$$

where for *c.c* we denoted the complex conjugated of the first integral in the square brackets. After a bit of algebra we achieve for the final form of $I_1(\vec{q})$:

$$\begin{cases} I_1(\vec{q}) = \frac{1}{ia\sqrt{3}}[f(x) - f(x^*)] \\ f(x) = \cos(|\vec{q}|x)\left[Si(|\vec{q}|x) + \frac{\pi}{2}\right] - \sin(|\vec{q}|x)Ci(|\vec{q}|x) \end{cases} \qquad \text{(A11)}$$

The integral $I_2(\vec{q})$ can be manipulated to get

$$I_2(\vec{q}) = \frac{1}{i\sqrt{3}} \left\{ \frac{1}{(a+x)} \left[\int_{-x}^\infty dy \frac{\sin(|\vec{q}|(y+x))}{y} - \int_a^\infty dy \frac{\sin(|\vec{q}|(y-a))}{y} \right] - c.c. \right\}, \qquad \text{(A12)}$$

where *c.c.* this time represents the complex conjugate of the whole expression in the curly brackets. According to the same procedure used for the former integral $I_1(\vec{q})$ we achieve

$$\begin{cases} I_2(\vec{q}) = \frac{g(a,x)}{i\sqrt{3}(a+x)} - \frac{g(a,x^*)}{i\sqrt{3}(a+x^*)} \\ g(a,x) = f(x) + \cos(|\vec{q}|a)\left[Si(|\vec{q}|a) - \frac{\pi}{2}\right] - \\ \qquad\qquad - \sin(|\vec{q}|a)Ci(|\vec{q}|a). \end{cases} \qquad \text{(A13)}$$

We recover immediately the asymptotic expression $\Lambda(\vec{q}) \sim \frac{2\pi^{d/2}}{\Gamma(d/2)} \ln\left(\frac{1}{|\vec{q}|a}\right)$ by expanding Eqs(A4),(A6) and the solution of (A8) for small $|\vec{q}|$.

Appendix B: Fox function appearance

The Fox functions are defined as [33–35, 41]

$$H_{pq}^{mn}\left[y \left| \begin{array}{ccc} (a_1, A_1) & ... & (a_p, A_p) \\ (b_1, B_1) & ... & (b_q, B_q) \end{array} \right. \right] = \frac{1}{2\pi i} \int_L \chi(s) y^{-s} ds \qquad \text{(B1)}$$

with $1 \le m \le q$, $0 \le n \le p$. $\chi(s)$ represents the Mellin transform which takes the following form

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^p \Gamma(a_j + A_j s)}. \qquad \text{(B2)}$$

where A_j and B_j are positive numbers while a_j and b_j are complex. Empty products are interpreted as being unity.

In the expression (26) the function appearing in the denominator can be expressed as a Fox function, see eq.(27). Indeed it is sufficient to notice that the Mellin transform of $\frac{1}{1+y^\gamma}$ is given by

$$\chi(s) = \int_0^\infty \frac{1}{1+y^\gamma} y^{s-1} ds = \frac{\Gamma(s/\gamma)\Gamma(1 - s/\gamma)}{\gamma} \qquad \text{(B3)}$$

which matches the definition (B2) if and only if $m = n = p = q = 1$, $a_1 = b_1 = 0$ and $A_1 = B_1 = 1/\gamma$.

Appendix C: Fox function properties

In this Section we enumerate the properties of the Fox function (B1) that we use throughout our analysis. This list is not an exhaustive compendium of the Fox functions properties, for which the reader could refer to [34, 35, 41]. For convenience in this Section we adopt the following short notation

$$H_{pq}^{mn}\left[y \left| \begin{array}{ccc} (a_1, A_1) & ... & (a_p, A_p) \\ (b_1, B_1) & ... & (b_q, B_q) \end{array} \right. \right] = H_{pq}^{mn}\left[y \left| \begin{array}{c} [a_p, A_p] \\ [b_q, B_q] \end{array} \right. \right]. \qquad \text{(C1)}$$

The useful rules are hereafter listed:

$$H_{p\,q}^{m\,n}\left[y\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=H_{q\,p}^{n\,m}\left[\frac{1}{y}\left|\begin{array}{c}[1-b_q,B_q]\\ [1-a_p,A_p]\end{array}\right.\right] \quad \text{(C2)}$$

$$y^{\sigma}H_{p\,q}^{m\,n}\left[y\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=H_{p\,q}^{m\,n}\left[y\left|\begin{array}{c}[a_p+\sigma A_p,A_p]\\ [b_q+\sigma B_q,B_q]\end{array}\right.\right] \quad \text{(C3)}$$

$$\int_0^{\infty}dy\,y^{\alpha-1}J_{\nu}(\sigma y)H_{p\,q}^{m\,n}\left[\omega y^r\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=$$

$$\frac{2^{\alpha-1}}{\sigma^{\alpha}}H_{p+2,q}^{m,n+1}\left[\omega\left(\frac{2}{\sigma}\right)^r\left|\begin{array}{c}\left(1-\frac{\alpha+\nu}{2},\frac{r}{2}\right)[a_p,A_p]\left(1-\frac{\alpha-\nu}{2},\frac{r}{2}\right)\\ [b_q,B_q]\end{array}\right.\right] \quad \text{(C4)}$$

$$\int_0^{\infty}dy\,y^{\alpha-1}\cos(\sigma y)H_{p\,q}^{m\,n}\left[\omega y^r\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=$$

$$\frac{2^{\alpha-1}\sqrt{\pi}}{\sigma^{\alpha}}H_{p+2,q}^{m,n+1}\left[\omega\left(\frac{2}{\sigma}\right)^r\left|\begin{array}{c}\left(\frac{2-\alpha}{2},\frac{r}{2}\right)[a_p,A_p]\left(\frac{1-\alpha}{2},\frac{r}{2}\right)\\ [b_q,B_q]\end{array}\right.\right] \quad \text{(C5)}$$

$$y^r\frac{d^r}{dy^r}H_{p\,q}^{m\,n}\left[y^{\delta}\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=H_{p+1,q+1}^{m,n+1}\left[y^{\delta}\left|\begin{array}{c}(0,\delta)[a_p,A_p]\\ [b_q,B_q](r,\delta)\end{array}\right.\right] \quad \text{(C6)}$$

$$H_{p\,q}^{m\,n}\left[y\left|\begin{array}{c}[a_p,A_p]\\ [b_q,B_q]\end{array}\right.\right]=\sum_{i=1}^m\sum_{k=0}^{\infty}c_{ik}\frac{(-1)^k}{k!B_i}y^{\frac{b_i+k}{B_i}}$$

$$c_{ik}=\frac{\prod_{j=1,j\neq i}^m\Gamma\left(b_j-\frac{(b_i+k)B_j}{B_i}\right)\prod_{j=1}^n\Gamma\left(1-a_j+\frac{(b_i+k)A_j}{B_i}\right)}{\prod_{j=m+1}^q\Gamma\left(1-b_j+\frac{(b_i+k)B_j}{B_i}\right)\prod_{j=n+1}^p\Gamma\left(a_j-\frac{(b_i+k)A_j}{B_i}\right)} \quad \text{(C7)}$$

This expansion is valid whenever $\sum_{i=1}^q B_i-\sum_{i=1}^p A_i\geq 0$ or $\sum_{i=1}^q B_i-\sum_{i=1}^p A_i<0$ and $\sum_{i=1}^n A_i-\sum_{i=n+1}^p A_i+sum_{i=1}^m B_i-sum_{i=m+1}^q B_i>0$. Empty products are interpreted as being unity.

Appendix D: Static correlator for fluid membranes

In this appendix we show how to get a general expression for the static correlator (48). Indeed using the property (C6), eq.(48) becomes

$$\left\langle\left[h(\vec{x},0)-h\left(\vec{x'},0\right)\right]^2\right\rangle=\frac{k_BT}{2\pi}\left|\vec{x}-\vec{x'}\right|^2\left\{0.25-\right.$$

$$\left.-\frac{1}{8\pi}\int_0^{+\infty}\frac{dy}{y^3}H_{2\,4}^{2\,2}\left[y\left|\begin{array}{cccc}(0,1)&\left(\frac{5}{6},\frac{1}{6}\right)\\ \left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)&\left(0,\frac{1}{2}\right)&(1,1)\end{array}\right.\right]\right\}. \quad \text{(D1)}$$

Introducing the lower cut-off $y_0=\left|\vec{x}-\vec{x'}\right|/L$ and extracting the logarithmic term we can recast the previous expression as

$$\left\langle\left[h(\vec{x},0)-h\left(\vec{x'},0\right)\right]^2\right\rangle=$$

$$\frac{k_BT}{4\pi}\left|\vec{x}-\vec{x'}\right|^2\left\{\ln\left(\frac{L}{|\vec{x}-\vec{x'}|}\right)+0.5-\frac{1}{4\pi}\int_{|\vec{x}-\vec{x'}|/L}^1\frac{dy}{y^3}\times\right.$$

$$\times\left(4\pi y^2+H_{2\,4}^{2\,2}\left[y\left|\begin{array}{cccc}(0,1)&\left(\frac{5}{6},\frac{1}{6}\right)\\ \left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)&\left(0,\frac{1}{2}\right)&(1,1)\end{array}\right.\right]\right)-$$

$$\left.-\frac{1}{4\pi}\int_1^{\infty}\frac{dy}{y^3}H_{2\,2}^{2\,4}\left[y\left|\begin{array}{cccc}(0,1)&\left(\frac{5}{6},\frac{1}{6}\right)\\ \left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)&\left(0,\frac{1}{2}\right)&(1,1)\end{array}\right.\right]\right\}. \quad \text{(D2)}$$

The last term in the brackets is transformed with the use of (C2) and (C3), getting finally

$$\left\langle\left[h(\vec{x},0)-h\left(\vec{x'},0\right)\right]^2\right\rangle=$$

$$\frac{k_BT}{4\pi}\left|\vec{x}-\vec{x'}\right|^2\left\{\ln\left(\frac{L}{|\vec{x}-\vec{x'}|}\right)+0.5-\frac{1}{4\pi}\int_{|\vec{x}-\vec{x'}|/L}^1\frac{dy}{y^3}\times\right.$$

$$\times\left(4\pi y^2+H_{2\,4}^{2\,2}\left[y\left|\begin{array}{cccc}(0,1)&\left(\frac{5}{6},\frac{1}{6}\right)\\ \left(0,\frac{1}{2}\right)&\left(\frac{5}{6},\frac{1}{6}\right)&\left(0,\frac{1}{2}\right)&(1,1)\end{array}\right.\right]\right)-$$

$$\left.-\frac{1}{4\pi}\int_0^1 dyH_{4\,2}^{2\,2}\left[y\left|\begin{array}{cccc}\left(\frac{3}{2},\frac{1}{2}\right)&\left(\frac{1}{3},\frac{1}{6}\right)&\left(\frac{3}{2},\frac{1}{2}\right)&(1,1)\\ (2,1)&\left(1,\frac{5}{6}\right)\end{array}\right.\right]\right\}. \quad \text{(D3)}$$

Expanding the H-functions in the former expressions by use of (C7) we can then get the corrections to 0.5 in the brackets.

Appendix E: Generalized Langevin equation for $\Delta_{DA}(t)$

In this appendix we derive the generalized Langevin equation (GLE) for the single component of the donor-acceptor distance $\Delta_{D-A}(t)$ in a protein, and then evaluate its autocorrelation function as arising from the outlined framework. The derivation of the GLE, as well as the corresponding correlation function, traces that proposed in [19] for single file systems: the main difference is that in our analysis we will make use of the Fourier transform instead of Laplace in the time domain.

Subtracting the FLE (21) for the donor position $h(x_D,t)$ from the FLE for the acceptor $h(x_A,t)$, the corresponding FLE for $\Delta_{D-A}(t)$ is achieved:

$$2D_+^{1/2}\Delta_{D-A}(t)=\zeta_{D-A}(t) \quad \text{(E1)}$$

where we implicitly assumed that $x_D<x_A$ along the protein backbone, and $\zeta_{D-A}(t)=\zeta(x_A,t)-\zeta(x_D,t)$. Eq.(E1) does not satisfy the generalized FD relation, as it is straightforwardly shown by calculating the correlation function of the noise

$$\langle\zeta_{D-A}(t)\zeta_{D-A}(t')\rangle=$$
$$2\left(\langle\zeta(x_A,t)\zeta(x_A,t')\rangle-\langle\zeta(x_A,t)\zeta(x_D,t')\rangle\right). \quad \text{(E2)}$$

The second term in the RHS of the former expression can be derived using the general formula (33) or by direct calculation, once one notices that in this simple case the function $\Phi(x,\omega)$ appearing in (18) is

$$\Phi(x,\omega) = e^{-|x|\sqrt{-i\omega}} \tag{E3}$$

As a matter of fact, in the ω space the FD relation reads [48]

$$\langle \zeta(\omega)\zeta(\omega') \rangle = 4\pi k_B T \, \Re e \left[\gamma(\omega) \right] \delta(\omega + \omega') \tag{E4}$$

where $\gamma(\omega)$ represents the Fourier transform of the damping kernel , which in (E1) is given by

$$\gamma_{D-A}(\omega) = \frac{2}{\sqrt{-i\omega}}, \tag{E5}$$

and, on the other hand,

$$\langle \zeta_{D-A}(\omega)\zeta_{D-A}(\omega') \rangle = \frac{8\pi k_B T\sqrt{2}}{|\omega|^{1/2}} \left[1 - e^{-\chi_{D-A}(\omega)} \left(\cos\chi_{D-A}(\omega) - \sin\chi_{D-A}(\omega) \right) \right] \tag{E6}$$

with $\chi_{D-A}(\omega) = \frac{|\omega|^{1/2}(x_A - x_D)}{\sqrt{2}}$.

To restore the validity of the FD relation we multiply both the terms of (E1) by

$$C(\omega) = \frac{1 - e^{\sqrt{-i\omega}(x_a - x_D)}}{2 \left(1 + e^{-2\chi_{D-A}(\omega)} - 2e^{-\chi_{D-A}(\omega)} \cos\chi_{D-A}(\omega) \right)} \tag{E7}$$

obtaining the following form of the GLE in the Fourier space

$$\tilde{\gamma}_{D-A}(\omega)(-i\omega)\Delta_{D-A}(\omega) = \tilde{\zeta}_{D-A}(\omega) \tag{E8}$$

with $\tilde{\gamma}_{D-A}(\omega) = \frac{1}{\sqrt{-i\omega}\left(1 - e^{\sqrt{-i\omega}(x_a - x_D)}\right)}$ and $\tilde{\zeta}_{D-A}(\omega) = C(\omega)\zeta_{D-A}(\omega)$. Now, it is clear that for small ω the Fourier transform of the damping kernel gets to the asymptotic value

$$\tilde{\gamma}_{D-A}(\omega) \rightarrow \frac{1}{(-i\omega)(x_A - x_D)}.$$

Therefore we sum and subtract on the RHS of (E8) the asymptotic expression $\frac{1}{(-i\omega)(x_A - x_D)}$, obtaining

$$\tilde{\gamma}_{D-A}^{eff}(\omega)(-i\omega)\Delta_{D-A}(\omega) = -\omega_0^2\Delta_{D-A}(\omega) + \tilde{\zeta}_{D-A}(\omega) \tag{E9}$$

where $\tilde{\gamma}_{D-A}^{eff}(\omega) = \tilde{\gamma}_{D-A}(\omega) - \frac{1}{(-i\omega)(x_A - x_D)}$ and $\omega_0^2 = \frac{1}{x_A - x_D}$.

It is immediate to show that the FD relation (E4) still holds and inverting in time domain we finally get the sought form of the GLE for the donor-acceptor distance which tends to an FLE with fractional derivative of order $1/2$ in the long time limit [11, 19, 26, 27]

$$\int_{-\infty}^t \tilde{\gamma}_{D-A}^{eff}(t-t')\frac{d}{dt'}\Delta_{D-A}(\omega) = -\omega_0^2\Delta_{D-A}(t) + \tilde{\zeta}_{D-A}(t) \tag{E10}$$

We now want to evaluate the donor-acceptor autocorrelation function: we firstly need the expression of $\Delta_{D-A}(\omega)$ from (E9), i.e.

$$\Delta_{D-A}(\omega) = \frac{\tilde{\zeta}_{D-A}(\omega) \left(1 - e^{\sqrt{-i\omega}(x_a - x_D)} \right)}{\sqrt{-i\omega}} \tag{E11}$$

from whom, thanks to the noise autocorrelation function

$$\langle \tilde{\zeta}_{D-A}(\omega)\tilde{\zeta}_{D-A}(\omega') \rangle = \frac{2\pi k_B T\sqrt{2}}{|\omega|^{1/2}} \frac{1 - e^{-\chi_{D-A}(\omega)}(\cos\chi_{D-A}(\omega) - \sin\chi_{D-A}(\omega))}{1 + e^{-2\chi_{D-A}(\omega)} - 2e^{-\chi_{D-A}(\omega)}\cos\chi_{D-A}(\omega)}, \tag{E12}$$

we derive the correlation function

$$\langle \Delta_{D-A}(\omega)\Delta_{D-A}(\omega') \rangle = \frac{2\pi k_B T\sqrt{2}}{|\omega|^{3/2}} \times$$
$$\times \left[1 - e^{-\chi_{D-A}(\omega)} \left(\cos\chi_{D-A}(\omega) - \sin\chi_{D-A}(\omega) \right) \right] \delta(\omega + \omega') \tag{E13}$$

Hence inverting in time the previous expression and changing variable ($y = |\omega|^{1/2}(x_A - x_D)/2$) we finally get

$$\langle \Delta_{D-A}(t)\Delta_{D-A}(t') \rangle = \frac{\sqrt{2}k_B T}{\pi}(x_A - x_D) \times$$
$$\times \int_0^\infty dy \frac{\cos\left(\frac{4y^2|t-t'|}{(x_A - x_D)^2} \right)}{y^2} \left[1 - e^{-\sqrt{2}y} \left(\cos\sqrt{2}y - \sin\sqrt{2}y \right) \right] \tag{E14}$$

which exactly matches the expression furnished in (56).

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[47] According to (13) the original formulation of the Edward-Wilkinson model cannot be taken since for $z = 2$ and $d = 2$ (see [4]) the asymptotic diffusion is logarithmic as already stressed in [44], in this case we then assume $z = 1$.

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